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$L_q(L_p)$ theory and Hölder estimates for parabolic SPDEs

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Abstract

Hölder estimates are given for the solutions of parabolic stochastic partial differential equations in C^1 domains. Also existence and uniqueness theorems are presented in L_p-spaces with weights. It is allowed that the powers of summability with respect to space and time to be different.

MSC: 60H15; 35R60

Keywords: Stochastic partial differential equations; Hölder estimate

1. Introduction

Let G be an open set in $\mathbb{R}^d$ with C^1 boundary. We consider parabolic stochastic partial differential equations (SPDEs) of the forms

$$du = (a^{ij}u_{x^ix^j} + b^iu_{x^i} + cu + f)\,dt + (v^ku + g^k)\,dw_t^k, \tag{1.1}$$

$$du = (D_i(a^{ij}u_{x^i} + b^iu + f^i) + \bar{b}^iu_{x^i} + cu + f)\,dt + g^k\,dw_t^k, \tag{1.2}$$

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given for $x \in G, t \geqslant 0$. Here w_t^k are independent one-dimensional Wiener processes, i and j go from 1 to d, and k runs through $\{1, 2, \ldots\}$. The coefficients a^{ij}, b^i, $\bar{b}^i$, c, v^k and the free terms f^i, f, g^k are random functions depending on $t > 0$ and $x \in G$.

Our approach is based on Sobolev spaces and we show that if

$$f^i, f, |g|_{\ell_2} \in L_q(\Omega \times [0, T], L_p(G)), \tag{1.3}$$

then the solutions of Eqs. (1.1) and (1.2) satisfy

$$E|u|^q_{C^\alpha(G \times [0,T])} < \infty \tag{1.4}$$

with α depending on p and q. In particular, if $\kappa_0 := 1 - 2/q - d/p > 0$, then for any $\kappa \in (0, \kappa_0)$ the solutions of (1.1) satisfy

$$E|\sup_{t \leqslant T} \sup_{x,y \in G} |u(t,x) - u(t,y)|/|x-y|^\kappa|^q < \infty, \tag{1.5}$$

$$E|\sup_{x \in G} \sup_{t,s \leqslant T} |u(t,x) - u(s,x)|/|t-s|^{\kappa/2}|^q < \infty. \tag{1.6}$$

Observe that (1.5) and (1.6) hold for any $\kappa \in (0, 1)$ if (1.3) is satisfied for all $p, q > 0$.

The motivation of this paper comes from [10], where the equations

$$\mathrm{d}u = L_i u \,\mathrm{d}t + g \,\mathrm{d}w_t, \quad i = 1, 2, \tag{1.7}$$

$$L_1 u = a^{ij} u_{x^i x^j} + b^i u_{x^i} + cu, \quad L_2 u = D_i(a^{ij} u_{x^j} + b^i u) + \bar{b}^i u_{x^i} + cu$$

are considered in C^∞ domain. Approaches in [10] are based on the theory of semigroup (see, for instance, [2]), and it is assumed that the coefficients are *nonrandom* and *independent* of time. Also to get estimates like (1.4), it is additionally assumed that the coefficients of the operator L_1 are smooth function of x and (instead of (1.3))

$$\sup_{\omega,t} \|g\|_{L_p(G)} < \infty \quad \left(\text{or } \sup_{\omega,t,x} |g| < \infty\right). \tag{1.8}$$

In this paper we rediscover all the main results in [10], but we impose only minimal regularity conditions on the coefficients $a^{ij}, b^i, \bar{b}^i, c, v^k$ and free terms f^i, f, g. For instance, coefficients of our equations are measurable functions of ω, t, x and are possibly unbounded.

It is worth mentioning that in [4] the author obtained Hölder estimates like (1.5) and (1.6) (with $p = q$) for the solutions of the equation

$$\mathrm{d}u = (D_i(a^{ij} u_{x^i} + b^i u + f^i) + \bar{b}^i u_{x^i} + cu + f)\,\mathrm{d}t + (\sigma^{ik} u_{x^i} + v^k u + g^k)\,\mathrm{d}w_t^k,$$

where the leading coefficients a^{ij} and σ^{ik} are assumed to be pointwise continuous in x. In this article (even though we are assuming that $v^k = \sigma^{ik} = 0$) we do not assume the continuity of the coefficients in Eq. (1.2), and we also consider the case $p \neq q$.

Our main results are stated in Section 2 and consist of Theorems 2.4 and 2.6. The proof of Theorem 2.4 depends on Theorem 2.6 and Theorem 2.6 is a particular result of the $L_q(L_p)$-theory developed in Section 3.

We finish the introduction with some notations. As usual $\mathbb{R}^d$ stands for the Euclidean space of points $x=(x^1,\dots,x^d)$, $\mathbb{R}^d_+=\{x\in\mathbb{R}^d : x^1>0\}$ and $B_r(x)=\{y\in\mathbb{R}^d : |x-y|<r\}$. For $i=1,\dots,d$, multi-indices $\alpha=(\alpha_1,\dots,\alpha_d)$, $\alpha_i\in\{0,1,2,\dots\}$, and functions $u(x)$ we set

$$u_{x^i}=\partial u/\partial x^i=D_iu,\quad D^\alpha u=D_1^{\alpha_1}\cdot\ldots\cdot D_d^{\alpha_d}u,\quad |\alpha|=\alpha_1+\cdots+\alpha_d.$$

2. Main results

Let $(\Omega,\mathscr{F},P)$ be a complete probability space, $\{\mathscr{F}_t,t\geqslant 0\}$ be an increasing filtration of σ-fields $\mathscr{F}_t\subset\mathscr{F}$, each of which contains all $(\mathscr{F},P)$-null sets. By $\mathscr{P}$ we denote the predictable σ-field generated by $\{\mathscr{F}_t,t\geqslant 0\}$ and we assume that on Ω we are given independent one-dimensional Wiener processes $w_t^1,w_t^2,\dots$, each of which is a Wiener process relative to $\{\mathscr{F}_t,t\geqslant 0\}$.

Assumption 2.1. The domain G is of class C^1_u. In other words, there exist constants $r_0,K_0>0$ such that for any $x_0\in\partial G$ there exists a one-to-one continuously differentiable mapping Ψ from $B_{r_0}(x_0)$ onto a domain $J\subset\mathbb{R}^d$ such that

(i) $J_+:=\Psi(B_{r_0}(x_0)\cap G)\subset\mathbb{R}^d_+$ and $\Psi(x_0)=0$;
(ii) $\Psi(B_{r_0}(x_0)\cap\partial G)=J\cap\{y\in\mathbb{R}^d : y^1=0\}$;
(iii) $\|\Psi\|_{C^1(B_{r_0}(x_0))}\leqslant K_0$ and $|\Psi^{-1}(y_1)-\Psi^{-1}(y_2)|\leqslant K_0|y_1-y_2|$ for any $y_i\in J$;
(iv) $|\Psi_x(x_1)-\Psi_x(x_2)|\leqslant\delta_0(|x_1-x_2|)$ for any $x_i\in B_{r_0}(x_0)$, where δ_0 a nondecreasing function defined on $[0,\infty)$ such that $\delta_0(\varepsilon)\downarrow 0$ as $\varepsilon\to 0$.

Assumption 2.2.

(i) The coefficients a^{ij}, b^i, $\bar{b}^i$, c and v^k are $\mathscr{P}\otimes\mathscr{B}(G)$-measurable functions.
(ii) There exist constants $\delta,K\in(0,\infty)$ such that for any x,t,ω and $\lambda\in\mathbb{R}^d$,

$$\delta|\lambda|^2\leqslant a^{ij}\lambda^i\lambda^j\leqslant K|\lambda|^2. \tag{2.1}$$

(iii) $p\in[2,\infty)$ and $q\geqslant p$.
(iv) $f^i,f,g\in\mathbb{L}^q_{p,d}(G,T):=L_q(\Omega\times[0,T],\mathscr{P},L_p(G))$, where for instance

$$\|g\|^q_{\mathbb{L}^q_{p,d}(G,T)}:=E\int_0^T\||g|_{\ell_2}\|^q_{L_p(G)}\,\mathrm{d}s.$$

First consider the equation

$$\mathrm{d}u=(D_i(a^{ij}u_{x^j}+b^iu+f^i)+\bar{b}^iu_{x^i}+cu+f)\,\mathrm{d}t+\sum_{k=1}^\infty g^k\,\mathrm{d}w_t^k,$$

$$u(0,\cdot)=0,\quad u|_{\partial G}=0. \tag{2.2}$$

Definition 2.3. A generalized solution of (2.2) is a function $u \in L_2(\Omega \times [0,T], \mathscr{P}; H^1_2(G))$, where $H^1_2(G) = \{u : u, u_x \in L_2(G)\}$, such that for any $\phi \in C_0^\infty(G)$ the equality

$$(u(t,\cdot),\phi) = -\int_0^t (a^{ij}u_{x^j} + b^i u + f^i, \phi_{x^i})\,\mathrm{d}s + \int_0^t (\bar{b}^i u_{x^i} + cu + f, \phi)\,\mathrm{d}s + \sum_{k=1}^\infty \int_0^t (g^k,\phi)\,\mathrm{d}w_s^k$$

holds for all $t \leqslant T$ (a.s).

Here is our first main result. Below $\alpha_0 \in (0,1)$ is a constant depending only on d, p, q, δ, K, T, G, and the constant N depends only on α and the data determining α_0.

Theorem 2.4. *Let the domain G be bounded and the above assumptions be satisfied. Assume*

$$|b^i(t,x)| + |\bar{b}(t,x)| + |c(t,x)| \leqslant K \quad \forall \omega, t, x. \tag{2.3}$$

Then there exists a unique generalized solution u of equation (2.2), *and furthermore if*

$$1 - 2/q - d/p =: \kappa_0 > 0, \quad p > 2, \tag{2.4}$$

then there exists α_0 such that for any $\alpha < \alpha_0$

$$E|u|^q_{C^\alpha(G\times[0,T])} \leqslant N\left(\sum_i \|f^i\|^q_{\mathbb{L}^q_{p,d}(G,T)} + \|f\|^q_{\mathbb{L}^q_{p,d}(G,T)} + \|g\|^q_{\mathbb{L}^q_{p,d}(G,T)}\right). \tag{2.5}$$

In particular, if

$$p > d \quad \text{and} \quad f^i, f, g \in L_q(\Omega \times [0,T], \mathscr{P}, L_p(G)) \quad \forall q > 0,$$

then one can fix $\bar{\alpha} > 0$ such that for any $k \geqslant 0$ and $\alpha < \bar{\alpha}$

$$E|u|^k_{C^\alpha(G\times[0,T])} < \infty. \tag{2.6}$$

Next we consider the equation

$$\mathrm{d}u = (a^{ij}u_{x^ix^j} + b^i u_{x^i} + cu + f)\,\mathrm{d}t + \sum_{k=1}^\infty (v^k u + g^k)\,\mathrm{d}w_t^k, \quad u(0,\cdot) = 0. \tag{2.7}$$

We say that u is a solution of (2.7) if u satisfies (2.7) in the sense of distribution (see (3.5)). Denote $\rho(x) := \operatorname{dist}(x, \partial G)$, and for $v \in \mathbb{R}, k = 0, 1, \ldots$, define

$$[f]^{(v)}_k = \sup_{\substack{x\in G\\|\beta|=k}} \rho^{k+v}(x)|D^\beta f(x)|, \quad |f|^{(v)}_k = \sum_{j=0}^k [f]^{(v)}_j.$$

Similarly we define $|\cdot|_k$ by dropping ρ in the above definitions. For $\sigma \in (0,1]$, denote $|f|_{C(G)} = \sup_x |f(x)|$ and

$$[f]_{C^\sigma(G)} = \sup_{x,y\in G} \frac{|f(x)-f(y)|}{|x-y|^\sigma}, \quad |f|_{C^\sigma(G)} = |f|_{C(G)} + [f]_{C^\sigma(G)}.$$

Also we use the same notations for ℓ_2-valued functions.

Assumption 2.5.

(i) The coefficient a^{ij} is uniformly continuous in G. In other words, for any $\varepsilon > 0$, there exists $\kappa = \kappa(\varepsilon)$ such that

$$|a^{ij}(t,x) - a^{ij}(t,y)| \leqslant \varepsilon,$$

whenever $t \geqslant 0, \omega \in \Omega$ and $x, y \in G$ with $|x-y| \leqslant \kappa$.

(ii) $\rho b^i, \rho^2 c, \rho \nu$ are bounded and

$$\lim_{\rho(x)\to 0} \sup_{\omega,t} [\rho(x)|b^i(x)| + \rho^2(x)|c(x)| + \rho(x)|\nu|_{\ell_2}] = 0.$$

(iii) For each ω, t,

$$|a|_1^{(0)} + |b|_1^{(1)} \leqslant \bar{K}.$$

The following theorem is a particular result of Theorem 3.4 (also see Remarks 3.3 and 3.5).

Theorem 2.6. *Let the domain G be bounded and Assumptions* 2.1, 2.2, 2.5 *be satisfied. Then there exists a unique solution u of the equation* (2.7) *such that $\rho^{-1}u, u_x \in L_q(\Omega \times [0,T], \mathscr{P}, L_p(G))$, and for this solution we have*

$$\|\rho^{-1}u\|^q_{\mathbb{L}^q_{p,d}(G,T)} + \|u_x\|^q_{\mathbb{L}^q_{p,d}(G,T)} \leqslant N(\|f\|^q_{\mathbb{L}^q_{p,d}(G,T)} + \|g\|^q_{\mathbb{L}^q_{p,d}(G,T)}). \tag{2.8}$$

Furthermore if $\kappa_0 = 1 - 2/q - d/p > 0$, then for any $\kappa \in (0, \kappa_0)$ the solution satisfies

$$\begin{aligned} &E|\sup_{t\leqslant\tau} \sup_{x,y\in G} |u(t,x) - u(t,y)|/|x-y|^\kappa|^q \\ &\quad + E|\sup_{x\in G} \sup_{t,s\leqslant\tau} |u(t,x) - u(s,x)|/|t-s|^{\kappa/2}|^q \\ &\quad \leqslant N_0(\|f\|^q_{\mathbb{L}^q_{p,d}(G,T)} + \|g\|^q_{\mathbb{L}^q_{p,d}(G,T)}), \end{aligned} \tag{2.9}$$

where the constant N_0 depends only on $d, p, q, \kappa, \delta, K, \bar{K}, T$ and G.

Remark 2.7. It is obvious that (i) if $f, g \in L_q(\Omega \times [0,T], \mathscr{P}, L_p(G))$ for all p and q, then (2.9) holds for any $\kappa \in (0,1)$, and (ii) if $p > d$ and $f, g \in L_q(\Omega \times [0,T], \mathscr{P}, L_p(G))$ for all q, then (2.9) holds for any $\kappa < 1 - d/p$.

Remark 2.8. We will see that the condition $f \in \mathbb{L}^q_{p,d}$ in Assumption 2.2 can be relaxed to the condition $f \in \mathbb{H}^{-1,q}_{p,d}(G,T)$ (this space will be introduced later).

Proof of Theorem 2.4. Since the uniqueness and existence of generalized solution is well known (see, for instance, [13,14]), we only show that there exists a solution u with properties stated in the theorem. For a measurable function h defined in $G \times [0,T]$, define $\|h\|_{p,q}$ from

$$\|h\|^q_{p,q} = \int_0^T \|h(t,\cdot)\|^q_{L_p(G)}\,\mathrm{d}t.$$

Step 1: Assume $g^k = 0$ for all k. Notice that the domain G is uniformly regular. In other words, there exist constants $\theta_0, \theta_1 > 0$ such that the inequality

$$\mathrm{mes}(B_\rho(x_0) \cap G) \leqslant (1-\theta_1)\mathrm{mes}B_\rho(x_0)$$

holds whenever $\rho \leqslant \theta_0, x_0 \in \partial G$. This, together with (2.1), (2.3) and (2.4), enables us to use Theorems 3.2.1, 3.4.2 and 3.10.1 in [11]. It follows that for almost all $\omega \in \Omega$, there exists a nonrandom solution $u^\omega \in L_2([0,T], H^1_2(G))$ of Eq. (2.2) such that u^ω is continuous up to the parabolic boundary and

$$\|u^\omega\|_{2,2} + \|u^\omega_x\|_{2,2} \leqslant N(\|f^i(\omega,\cdot,\cdot)\|_{p,q} + \|f(\omega,\cdot,\cdot)\|_{p,q}). \tag{2.10}$$

By Theorem A in [1],

$$\sup_{t,x} |u^\omega(t,x)| \leqslant N(\|f^i(\omega,\cdot,\cdot)\|_{p,q} + \|f(\omega,\cdot,\cdot)\|_{p,q}). \tag{2.11}$$

Also, by inspecting the proof of Theorem 3.10.1 in [11] one can easily check that there exist constants $\alpha > 0$ and N depending only d, p, q, δ, K, T and G such that

$$|u^\omega|_{C^\alpha(G\times[0,T])} \leqslant N(1 + \sup|u^\omega|). \tag{2.12}$$

Thus combining (2.11) and (2.12), one gets

$$|u^\omega|_{C^\alpha(G\times[0,T])} \leqslant N(1 + \|f^i(\omega,\cdot,\cdot)\|_{p,q} + \|f(\omega,\cdot,\cdot)\|_{p,q}). \tag{2.13}$$

For each $c > 0$, by considering cu^ω, cf^i, cf instead of u^ω, f^i, f respectively, one finds that (2.13) yields

$$|u^\omega|_{C^\alpha(G\times[0,T])} \leqslant N(1/c + \|f^i(\omega,\cdot,\cdot)\|_{p,q} + \|f(\omega,\cdot,\cdot)\|_{p,q})$$

and concludes

$$|u^\omega|_{C^\alpha(G\times[0,T])} \leqslant N(\|f^i(\omega,\cdot,\cdot)\|_{p,q} + \|f(\omega,\cdot,\cdot)\|_{p,q}). \tag{2.14}$$

Observe that for any fixed p, we may assume that $\alpha = \alpha(q)$ is a nondecreasing function of q. Indeed, if $q_1 \leqslant q_2 < \infty$, and all the conditions hold for $q = q_1$ and $q = q_2$, then for $\alpha = \alpha(q_1) \vee \alpha(q_2)$,

$$|u^\omega|_{C^\alpha(G\times[0,T])} \leqslant N(\|f^i(\omega,\cdot,\cdot)\|_{p,q_2} + \|f(\omega,\cdot,\cdot)\|_{p,q_2}).$$

Finally, define $u(\omega,t,x) = u^\omega(t,x)$, then it follows from (2.10) that u is a generalized solution of Eq. (2.2), where the measurability follows from the uniqueness result in [14]. Indeed, let $v \in L_2(\Omega \times [0,T], \mathscr{P}; H^1_2(G))$ be the solution of

Eq. (2.2). Observe that, for each ω, both $u^{\omega}(t,x)$ and $v(\omega,t,x)$ are determistic solutions of Eq. (2.2) and thus by the uniqueness of the solution, we get $u(\omega,\cdot) = v(\omega,\cdot)$ for each ω and thus $u = v \in L_2(\Omega \times [0,T], \mathscr{P}; H_2^1(G))$.

Step 2: By Theorem 2.6, there is a generalized solution v of the equation

$$\mathrm{d}v = (\Delta v + f)\,\mathrm{d}t + g^k\,\mathrm{d}w_t^k, \quad v(0,\cdot) = 0$$

such that v satisfies (2.8) and (2.9). Observe

$$\bar{f}^i, \bar{f} \in \mathbb{L}_p^q(G,T),$$

where $\bar{f}^i := (a^{ij} - \delta^{ij})v_{x^j} + b^i v + f^i$ and $\bar{f} := \bar{b}^i v_{x^i} + cv$.

By the results of Step 1, one can define a generalized solution $\bar{u}$ of

$$\mathrm{d}\bar{u} = (D_i(a^{ij}\bar{u}_{x^j} + b^i\bar{u} + \bar{f}^i) + \bar{b}^i\bar{u}_{x^i} + c\bar{u} + \bar{f})\,\mathrm{d}t$$

such that $\bar{u}$ satisfies

$$E|\bar{u}|^q_{C^\alpha(G\times[0,T])} \leqslant N(\|f^i\|^q_{\mathbb{L}_p^q(G,T)} + \|f\|^q_{\mathbb{L}_p^q(G,T)} + \|g\|^q_{\mathbb{L}_p^q(G,T)}),$$

where α is taken from (2.12). Finally observe that $u := v + \bar{u}$ satisfies (2.2) and all other assertions of the theorem follow. The theorem is proved. $\square$

3. SPDEs in $L_q([\![0,\tau]\!], \mathscr{P}, L_p)$ spaces

In this section we give uniqueness and solvability result of (1.1) in stochastic Banach spaces $\mathfrak{H}^{\gamma,q}_{p,\theta,0}(G,\tau)$ (see below for the notation).

As in [6] and [12], we use Banach spaces $H_p^\gamma, H_{p,\theta}^\gamma(G)$, where $\gamma,\theta \in \mathbb{R}$ and $p>1$. If n is a nonnegative integer, then

$$H_p^n = H_p^n(\mathbb{R}^d) = \{u : u, Du, \ldots, D^\alpha u \in L_p : |\alpha| \leqslant n\},$$

$$L_{p,\theta}(G) := H^0_{p,\theta}(G) = L_p(G, \rho^{\theta-d}\,\mathrm{d}x), \quad \rho(x) := \mathrm{dist}(x, \partial G),$$

$$H^n_{p,\theta}(G) := \{u : u, \rho u_x, \ldots, \rho^{|\alpha|}D^\alpha u \in L_{p,\theta}(G) : |\alpha| \leqslant n\}.$$

In general, by $H_p^\gamma = (1-\Delta)^{-\gamma/2}L_p$ we denote the space of Bessel potential

$$\|u\|_{H_p^\gamma} = \|(1-\Delta)^{\gamma/2}u\|_{L_p}$$

and the weighted Sobolev space $H^\gamma_{p,\theta}(G)$ is defined as set of all distributions u on G such that

$$\|u\|^p_{H^\gamma_{p,\theta}(G)} := \sum_{n=-\infty}^{\infty} \mathrm{e}^{n\theta}\|\zeta_{-n}(\mathrm{e}^n\cdot)u(\mathrm{e}^n\cdot)\|^p_{H_p^\gamma} < \infty, \tag{3.1}$$

where $\{\zeta_n : n \in \mathbb{Z}\}$ is a sequence of functions $\zeta_n \in C_0^\infty(G)$ such that

$$\sum_n \zeta_n \geqslant \mathrm{const} > 0, \quad |D^m\zeta_n(x)| \leqslant N(m)\mathrm{e}^{mn}.$$

If $G=\mathbb{R}^d_+$ we fix a function $\zeta \in C_0^\infty(\mathbb{R}_+)$ such that

$$\sum_{n\in\mathbb{Z}} \zeta(\mathrm{e}^{n+x})>0, \quad \forall x \in \mathbb{R} \tag{3.2}$$

and define $\zeta_n(x)=\zeta(\mathrm{e}^n x)$, then (3.1) becomes

$$\|u\|^p_{H^\gamma_{p,\theta}} := \sum_{n=-\infty}^{\infty} \mathrm{e}^{n\theta}\|\zeta(\cdot)u(\mathrm{e}^n\cdot)\|^p_{H^\gamma_p}<\infty. \tag{3.3}$$

It is known that the space $H^\gamma_{p,\theta}$ is independent of the choice ζ, and $H^\gamma_{p,\theta}(G)$ and its norm are independent of $\{\zeta_n\}$ if G is bounded.

For any stopping time τ, denote $⟦0,\tau⟧=\{(\omega,t): 0<t\leqslant\tau(\omega)\}$,

$$\mathbb{H}^{\gamma,q}_p(\tau)=L_q(⟦0,\tau⟧,\mathscr{P},H^\gamma_p), \quad \mathbb{H}^{\gamma,q}_{p,\theta}(G,\tau)=L_q(⟦0,\tau⟧,\mathscr{P},H^\gamma_{p,\theta}(G)),$$

$$\mathbb{L}^q_p(\tau)=\mathbb{H}^{0,q}_p(\tau), \quad \mathbb{L}^q_{p,\theta}(G,\tau)=\mathbb{H}^{0,q}_{p,\theta}(G,\tau).$$

Fix (see [5]) a bounded real-valued function ψ defined in $\bar{G}$ such that for any multi-index α,

$$\sup_G \rho^{|\alpha|}(x)|D^\alpha\psi_x(x)|<\infty$$

and the functions ψ and ρ are comparable in a neighborhood of ∂G.

By $\mathfrak{H}^{\gamma,q}_{p,\theta,0}(G,\tau)$ we denote the space of all functions $u\in\psi\mathbb{H}^{\gamma,q}_{p,\theta}(G,\tau)$ such that $u(0,\cdot)=0$ and for some $f\in\psi^{-1}\mathbb{H}^{\gamma-2,q}_{p,\theta}(G,\tau)$, $g\in\mathbb{H}^{\gamma-1,q}_{p,\theta}(G,\tau)$,

$$\mathrm{d}u=f\,\mathrm{d}t+g^k\,\mathrm{d}w^k_t, \tag{3.4}$$

in the sense of distributions. In other words, for any $\phi\in C_0^\infty(G)$, the equality

$$(u(t,\cdot),\phi)=\int_0^t (f(s,\cdot),\phi)\,\mathrm{d}s+\sum_0^\infty\int_0^t (g^k(s,\cdot),\phi)\,\mathrm{d}w^k_s \tag{3.5}$$

holds for all $t\leqslant\tau$ with probability 1.

The norm in $\mathfrak{H}^{\gamma,q}_{p,\theta,0}(G,\tau)$ is introduced by

$$\|u\|_{\mathfrak{H}^{\gamma,q}_{p,\theta}(G,\tau)}=\|\psi^{-1}u\|_{\mathbb{H}^{\gamma,q}_{p,\theta}(G,\tau)}+\|\psi f\|_{\mathbb{H}^{\gamma-2,q}_{p,\theta}(G,\tau)}+\|g\|_{\mathbb{H}^{\gamma-1}_{p,\theta}(G,\tau)}.$$

It is easy to check that the space $\mathfrak{H}^{\gamma,q}_{p,\theta,0}(G,\tau)$ and its norm are independent of the choice of ψ if G is bounded.

The proof of following lemma is given at the end of this section.

Lemma 3.1. *Let the domain G be bounded, $u\in\mathfrak{H}^{\gamma,q}_{p,\theta,0}(G,\tau)$, and*

$$2/q<\alpha<\beta\leqslant 1. \tag{3.6}$$

Then for any $0 \leqslant s < t \leqslant \tau$,

$$E\|\psi^{\beta-1}(u(t)-u(s))\|^q_{H^{\gamma-\beta}_{p,\theta}(G)} \leqslant N|t-s|^{q\beta/2-1}\|u\|^q_{\mathfrak{H}^{\gamma,q}_{p,\theta}(G,\tau)}, \tag{3.7}$$

$$E|\psi^{\beta-1}u|^q_{C^{\alpha/2-1/q}([0,\tau],H^{\gamma-\beta}_{p,\theta}(G))} \leqslant N\|u\|^q_{\mathfrak{H}^{\gamma,q}_{p,\theta}(G,\tau)}. \tag{3.8}$$

It is known that (see Theorem 4.3 in [12]) if $\gamma - d/p = k + v$ for some $k = 0, 1, 2, \ldots$, $v \in (0,1)$, then for any multi-indices i and j such that $|i| \leqslant k$ and $|j| = k$,

$$|\psi^{|j|+\theta/p}D^j u|_{C(G)} + [\psi^{|i|+v+\theta/p}D^i u]_{C^v(G)} \leqslant N\|u\|_{H^\gamma_{p,\theta}(G)}. \tag{3.9}$$

Thus (3.8) yields the following results.

Lemma 3.2. *Assume*

$$2/q < \alpha < \beta \leqslant 1, \quad \gamma - \beta - d/p = k + \varepsilon,$$

where $k = 0, 1, 2\ldots, \varepsilon \in (0,1)$. *Then for* $v := \beta - 1 + \theta/p$ *and multi-indices* i *and* j *such that* $|i| \leqslant k$ *and* $|j| = k$, *we have*

$$\begin{aligned} &E\sup_{t,s\leqslant\tau}|t-s|^{1-q\alpha/2}(|\psi^{v+|i|}D^i(u(t)-u(s))|^q_{C(G)} + [\psi^{v+|j|+\varepsilon}D^j(u(t)-u(s))]^q_{C^\varepsilon(G)}) \\ &\quad \leqslant N\|u\|^q_{\mathfrak{H}^{\gamma,q}_{p,\theta}(G,\tau)}. \end{aligned} \tag{3.10}$$

Remark 3.3. It follows from (3.10) that if $\theta = d$ and $\kappa_0 := 1 - 2/q - d/p > 0$, then for any $\kappa \in (0, \kappa_0)$, $u \in \mathfrak{H}^{1,q}_{p,\theta,0}(G,\tau)$,

$$E|\sup_{t\leqslant\tau}\sup_{x,y\in G}|u(t,x)-u(t,y)|/|x-y|^\kappa|^q \leqslant N\|u\|^q_{\mathfrak{H}^{1,q}_{p,\theta,0}(G,\tau)}, \tag{3.11}$$

$$E|\sup_{x\in G}\sup_{t,s\leqslant\tau}|u(t,x)-u(s,x)|/|t-s|^{\kappa/2}|^q \leqslant N\|u\|^q_{\mathfrak{H}^{1,q}_{p,\theta,0}(G,\tau)}. \tag{3.12}$$

Indeed, for (3.11) take $\beta = \kappa_0 - \kappa + 2/q$, then we have $\varepsilon = 1 - \beta - d/p = \kappa = -v$. To get (3.12), take $\alpha = \kappa + 2/q$, and note that $2/q < \alpha < 1 - d/p < 1$ and $1 - q\alpha/2 = -q\kappa/2$. Thus it is enough to choose β such that $\alpha < \beta < 1 - d/p$ because $v < 0$ and

$$|u(t)-u(s)|_{C(G)} \leqslant N|\psi^v(u(t)-u(s))|_{C(G)}.$$

From now on we assume

$$d - \chi < \theta < d - 1 + p, \quad \tau \leqslant T, \tag{3.13}$$

where $\chi = \chi(d,p,q,\delta) \in (0,1]$ is the constant taken from Theorem 3.2 in [8]. If $p = q$, then one can take $\chi = 1$.

Theorem 3.4. *Let the domain* G *be bounded and Assumptions* 2.1, 2.2(i)–(iii), 2.5 *be satisfied. Then, for any* $f \in \psi^{-1}\mathbb{H}^{-1,q}_{p,\theta}(G,\tau)$, $g \in \mathbb{L}^q_{p,\theta}(G,\tau)$ *Eq.* (2.7) *admits a unique* *solution* $u \in \mathfrak{H}^{1,q}_{p,\theta,0}(G,\tau)$. *Furthermore for this solution we have*

$$\|\psi^{-1}u\|_{\mathbb{H}^{1,q}_{p,\theta}(G,\tau)} \leqslant N(\|\psi f\|_{\mathbb{H}^{-1,q}_{p,\theta}(G,\tau)} + \|g\|_{\mathbb{L}^{q}_{p,\theta}(G,\tau)}). \tag{3.14}$$

Remark 3.5. It is well known (see Theorem 4.1 in [12]) that $\psi^{-1}v \in H^1_{p,\theta}(G)$ if and only if $\psi^{-1}v, v_x \in L_{p,\theta}(G)$, and moreover

$$\|\psi^{-1}v\|_{H^1_{p,\theta}(G)} \sim \|\psi^{-1}v\|_{L_{p,\theta}(G)} + \|v_x\|_{L_{p,\theta}(G)}.$$

To prove Theorem 3.4, we need the following results on $\mathbb{R}^d$ and $\mathbb{R}^d_+$. We write $u \in \mathscr{H}^{\gamma,q}_{p,0}(\tau)$ if $u(0,\cdot) = 0, u \in \mathbb{H}^{\gamma,q}_p(\tau)$ and u satisfies (3.4) for some $f \in \mathbb{H}^{\gamma-2,q}_p(\tau), g \in \mathbb{H}^{\gamma-1,q}_p(\tau)$. The norm in $\mathscr{H}^{\gamma,q}_{p,0}$ is introduced by

$$\|u\|^q_{\mathscr{H}^{\gamma,q}_p(\tau)} = \|u\|^q_{\mathbb{H}^{\gamma,q}_p(\tau)} + \|f\|^q_{\mathbb{H}^{\gamma-2,q}_p(\tau)} + \|g\|^q_{\mathbb{H}^{\gamma-1,q}_p(\tau)}.$$

Lemma 3.6. *Let* $G = \mathbb{R}^d$ *and Assumption* 2.2(i)–(iii) *be satisfied. Assume*

$$\sup_{\omega,t}(|a|_1 + |b|_1 + |c|_0 + |v|_0) \leqslant K_1,$$

$$\sup_{\omega,t}\sup_{x,y} |a^{ij}(t,x) - a^{ij}(t,y)| \leqslant \beta.$$

Then there exists $\beta_0 > 0$ *depending only on* d,p,q,δ *and* K (*thus independent of* K_1) *such that if* $\beta \leqslant \beta_0$, *then for any* $f \in \mathbb{H}^{-1,q}_p(\tau), g \in \mathbb{L}^q_p(\tau)$ *Eq.* (2.7) *admits a unique solution u in the class* $\mathscr{H}^{1,q}_{p,0}(\tau)$ *and furthermore*

$$\|u\|_{\mathbb{H}^{1,q}_p(\tau)} \leqslant N(\|f\|_{\mathbb{H}^{-1,q}_p(\tau)} + \|g\|_{\mathbb{L}^q_p(\tau)}), \tag{3.15}$$

where N depends only on d,p,q,δ,K,K_1 *and* T.

Proof. As usual we may assume $\tau \equiv T$. By Theorem 2.1 in [8], the results are true if a^{ij} is independent of x and other coefficients are zero. Thus considering the method of continuity, we only show that there exists β_0 such that the a priori estimate (3.15) holds true given that a solution $u \in \mathscr{H}^{1,q}_{p,0}(T)$ already exists and $\beta \leqslant \beta_0$.

Case 1: $b^i = c = v^k = 0$. We repeat the proof of Lemma 6.6 in [6]. Denote $a^{ij}_0(t,x) = a^{ij}(t,0)$. We rewrite the Eq. (2.7) as

$$\mathrm{d}u = (a^{ij}_0 + \bar{f})\,\mathrm{d}t + g^{ik}\,\mathrm{d}w^k_t, \tag{3.16}$$

where $\bar{f} = f + (a^{ij} - a^{ij}_0)u_{x^ix^j}$. By Theorem 2.1 in [8],

$$\|u\|_{\mathbb{H}^{1,q}_p(T)} \leqslant N(\|(a-a_0)u_{xx}\|_{\mathbb{H}^{-1,q}_p(T)} + \|f\|_{\mathbb{H}^{-1,q}_p(T)} + \|g\|_{\mathbb{H}^{1,q}_p(T)}), \tag{3.17}$$

and by Lemma 5.2 in [6],

$$\|(a-a_0)u_{xx}\|_{H^{-1}_p} \leqslant N|a-a_0|_{C^1}\|u_{xx}\|_{H^{-1}_p} \leqslant N|a-a_0|_{C^1}\|u\|_{H^1_p}.$$

Coming back to (3.17), we get

$$\|u\|_{\mathbb{H}^{1,q}_p(T)} \leqslant N_1 \sup_{\omega,t} |a-a_0|_{C^1}\|u\|_{\mathbb{H}^{1,q}_p(T)} + N\|f\|_{\mathbb{H}^{-1,q}_p(T)} + N\|g\|_{\mathbb{H}^{1,q}_p(T)},$$

where $N_1 = N_1(d,p,q,\delta,K)$. Choose β_0 such that for any $\beta \leqslant \beta_0$,

$$N_1\beta \leqslant \tfrac{1}{4}.$$

Next, observe that, for $a_m(t,x) := a(t/m^2, x/m)$ and $m \geqslant 1$, we have

$$N_1|a_m(t,\cdot) - a_m(t,0)|_{C^1} \leqslant \tfrac{1}{4} + N_1K_1/m.$$

Now fix m such that $N_1K_1/m \leqslant \frac{1}{4}$, then the claims of the lemma are true if we replace $u(t,x)$, a, w_t, f, g and T by $u(t/m^2, x/m)$, a_m, mw_{t/m^2}, $m^{-2}f(t/m^2, x/m)$, $m^{-1}g(t/m^2, x/m)$ and m^2T, respectively. After this it is enough to use the fact that the norms $\|\cdot\|_{H_p^\gamma}$ of $u(t/m^2, x/m)$ and $u(t,x)$ are comparable.

Case 2: In general, by the results of case 1, there exists β_0 such that if $\beta \leqslant \beta_0$, then for each $t \leqslant T$,

$$\|u\|^q_{\mathbb{H}^{1,q}_p(t)} \leqslant N(\|b^i u_{x^i} + cu + f\|^q_{\mathbb{H}^{-1,q}_p(t)} + \|v^k + g\|^q_{\mathbb{L}^q_p(T)}).$$

Here we use (see Lemma 5.2 in [6])

$$\|b^i u_x\|_{H_p^{-1}} + \|cu\|_{H_p^{-1}} + \|vu\|_{L_p} \leqslant N(|b|_1 + |c|_0 + |v|_0)\|u\|_{L_p}$$

and get for each $t \leqslant T$,

$$\|u\|^q_{\mathbb{H}^{1,q}_p(t)} \leqslant N\|u\|^q_{\mathbb{L}^q_p(t)} + N\|f\|^q_{\mathbb{H}^{-1,q}_p(T)} + N\|g\|^q_{\mathbb{L}^q_p(T)}. \tag{3.18}$$

By (3.18) and Theorem 4.11 in [9], for any $t \leqslant T$,

$$\|u\|^q_{\mathfrak{H}^{1,q}_p(t)} \leqslant N\int_0^t \|u\|^q_{\mathfrak{H}^{1,q}_p(s)}\,\mathrm{d}s + N\|f\|^q_{\mathbb{H}^{-1,q}_p(T)} + N\|g\|^q_{\mathbb{L}^q_p(T)}.$$

The estimate (3.15) follows from Gronwall's inequality. □

For the case $G = \mathbb{R}^d_+$, as in [8], by M^α we denote the operator of multiplying by $(x^1)^\alpha$ and $M = M^1$. Remember that we say $u \in \mathfrak{H}^{\gamma,q}_{p,\theta,0}$ if $u(0,\cdot) = 0$, $u \in M\mathbb{H}^{\gamma,q}_{p,\theta}$ and u satisfies (3.4) for some $f \in M^{-1}\mathbb{H}^{\gamma-2,q}_{p,\theta}$ and $g \in \mathbb{H}^{\gamma-1,q}_{p,\theta}$. The norm in $\mathfrak{H}^{\gamma,q}_{p,\theta,0}$ is introduced by

$$\|u\|^q_{\mathfrak{H}^{\gamma,q}_{p,\theta}(\tau)} = \|M^{-1}u\|^q_{\mathbb{H}^{\gamma,q}_{p,\theta}(\tau)} + \|Mf\|^q_{\mathbb{H}^{\gamma-2,q}_{p,\theta}(\tau)} + \|g\|^q_{\mathbb{H}^{\gamma-1,q}_{p,\theta}(\tau)}.$$

Lemma 3.7. *Let $G = \mathbb{R}^d_+$ and Assumption* 2.2(i)–(iii) *be satisfied. Also assume*

$$\sup_{\omega,t} |a|_1^{(0)} \leqslant \bar{K},$$

$$|a^{ij}(t,x) - a^{ij}(t,y)| + x^1|b(t,x)| + (x^1)^2|c(t,x)| + x^1|v(t,x)|_{\ell_2} \leqslant \beta,$$

whenever $\omega \in \Omega$, $t \geqslant 0$ and $x,y \in \mathbb{R}^d_+$. Then there exists $\bar{\beta}_0 > 0$ depending only on d,θ,p,q,δ,K and $\bar{K}$ such that if $\beta \leqslant \bar{\beta}_0$, then for any $f \in M^{-1}\mathbb{H}^{-1,q}_{p,\theta}(\tau)$, $g \in \mathbb{L}^q_{p,\theta}(\tau)$ Eq. (2.7) *admits a unique solution u in the class $\mathfrak{H}^{1,q}_{p,\theta,0}(\tau)$ and furthermore*

$$\|M^{-1}u\|_{\mathbb{H}^{1,q}_{p,\theta}(\tau)} \leqslant N(\|Mf\|_{\mathbb{H}^{-1,q}_{p,\theta}(\tau)} + \|g\|_{\mathbb{L}^q_{p,\theta}(\tau)}), \tag{3.19}$$

where N is independent of u,f,g and T.

Lemma 3.8. *Let all the assumptions of Lemma* 3.7 *be satisfied. Also assume that* $b^i = c = v^k = 0$. *Then there exists* $\beta' > 0$ *depending only on* $d, \theta, p, q, \delta, K$ *and* $\bar{K}$ *such that if* $\beta \leqslant \beta'$, *then for any* $f \in M^{-1}\mathbb{L}^q_{p,\theta}(\tau), g \in \mathbb{H}^{1,q}_{p,\theta}(\tau)$ *Eq.* (2.7) *admits a unique solution u in the class* $\mathfrak{H}^{2,q}_{p,\theta,0}(\tau)$ *and furthermore*

$$\|M^{-1}u\|_{\mathbb{H}^{2,q}_{p,\theta}(\tau)} \leqslant N(\|Mf\|_{\mathbb{L}^q_{p,\theta}(\tau)} + \|g\|_{\mathbb{H}^{1,q}_{p,\theta}(\tau)}), \tag{3.20}$$

where N is independent of u, f, g *and* T.

Proof. The results are true (see [8]) for any $\gamma \in \mathbb{R}$ if a^{ij} is independent of x and $b^i = c = v^k = 0$. Thus, we only show that the a priori estimate (3.20) holds given that a solution $u \in \mathfrak{H}^{2,q}_{p,\theta,0}(\tau)$ already exists.

Fix $x_0 \in \mathbb{R}^d_+$, and denote $a^{ij}_0(t,x) = a^{ij}(t,x_0)$. By Theorem 3.2 in [8] (rewrite the equation as in (3.16)),

$$\|M^{-1}u\|^q_{\mathbb{H}^{2,q}_{p,\theta}(\tau)} \leqslant N(\|M(a-a_0)u_{xx} + Mf\|^q_{\mathbb{L}^q_{p,\theta}(\tau)} + \|g\|^q_{\mathbb{H}^{1,q}_{p,\theta}(\tau)}).$$

Remember that for any $\gamma \in \mathbb{R}$ and $w \in MH^{\gamma+2}_{p,\theta}$ (see [7]),

$$\|Mw_{xx}\|_{H^\gamma_{p,\theta}} + \|w_x\|_{H^{\gamma+1}_{p,\theta}} \leqslant N\|M^{-1}w\|_{H^{\gamma+2}_{p,\theta}}. \tag{3.21}$$

Thus, if β is sufficiently small, then

$$\|M^{-1}u\|^q_{\mathbb{H}^{2,q}_{p,\theta}(\tau)} \leqslant N(\|Mf\|^q_{\mathbb{L}^q_{p,\theta}(\tau)} + \|g\|^q_{\mathbb{H}^{1,q}_{p,\theta}(\tau)}).$$

The lemma is proved. □

Proof of Lemma 3.7. *Step* 1: Assume that $b^i = c = v^k = 0$. In this case we prove the lemma directly without depending on an a priori estimate.

Take β' from Lemma 3.8 and assume

$$\beta \leqslant \beta'.$$

Then the operator $\mathscr{R}$ which maps the couples $(f,g) \in M^{-1}\mathbb{L}_{p,\theta}(\tau) \times \mathbb{H}^{1,q}_{p,\theta}(\tau)$ into the solutions $u \in \mathfrak{H}^{2,q}_{p,\theta,0}(\tau)$ of Eq. (2.7) is well-defined and bounded. Now take $(f,g) \in M^{-1}\mathbb{H}^{-1,q}_{p,\theta}(\tau) \times \mathbb{L}^q_{p,\theta}(\tau)$. By Corollary 2.12 in [7] we have the following representations:

$$f = MD_\ell f^\ell, \quad g^k = MD_\ell g^{\ell k}, \tag{3.22}$$

where $f^\ell \in M^{-1}\mathbb{L}^q_{p,\theta}, g^\ell \in \mathbb{H}^{1,q}_{p,\theta}, \ell = 1, 2, \ldots, d$ and

$$\sum_{\ell=1}^d \|Mf^\ell\|_{\mathbb{L}^q_{p,\theta}(\tau)} \leqslant N\|Mf\|_{\mathbb{H}^{-1,q}_{p,\theta}(\tau)}, \quad \sum_{\ell=1}^d \|g^\ell\|_{\mathbb{H}^{1,q}_{p,\theta}(\tau)} \leqslant N\|g\|_{\mathbb{L}^q_{p,\theta}(\tau)}. \tag{3.23}$$

Next denote $v^\ell = \mathscr{R}(f^\ell, g^\ell)$ and $\bar{v} = \sum_{\ell=1}^d MD_\ell v^\ell$. Then $\bar{v} \in M\mathbb{H}^1_{p,\theta}$ and satisfies

$$\mathrm{d}\bar{v} = (a^{ij}\bar{v}_{x^ix^j} + f + \bar{f})\,\mathrm{d}t + g^k\,\mathrm{d}w^k_t,$$

where

$$\bar{f} = v^{\ell}_{x^i x^j} M D_{\ell} a^{ij} - 2a^{i1} v^{\ell}_{x^{\ell} x^i}.$$

By assumptions one can easily check that

$$M\bar{f} \in \mathbb{L}^q_{p,\theta}(\tau).$$

Finally we define $\bar{u} = \mathscr{R}(\bar{f}, 0)$ and $u = \bar{v} - \bar{u}$. Then $u \in \mathfrak{H}^{1,q}_{p,\theta,0}(\tau)$ satisfies (2.7), and (3.19) follows from the formulas defining u. Indeed,

$$\begin{aligned}
\|M^{-1}\bar{v}\|_{\mathbb{H}^{1,q}_{p,\theta}(\tau)} &= \|v^{\ell}_{x^{\ell}}\|_{\mathbb{H}^{1,q}_{p,\theta}(\tau)} \leqslant N\|M^{-1}v^{\ell}\|_{\mathbb{H}^{2,q}_{p,\theta}(\tau)} \leqslant N\|Mf^{\ell}\|_{\mathbb{L}^q_{p,\theta}(\tau)} + N\|g^{\ell}\|_{\mathbb{H}^{1,q}_{p,\theta}(\tau)} \\
&\leqslant N\|Mf\|_{\mathbb{H}^{-1,q}_{p,\theta}(\tau)} + N\|g\|_{\mathbb{L}^q_{p,\theta}(\tau)}, \|M^{-1}\bar{u}\|_{\mathbb{H}^{1,q}_{p,\theta}(\tau)} \\
&\leqslant \|M^{-1}\bar{u}\|_{\mathbb{H}^{2,q}_{p,\theta}(\tau)} \leqslant N\|M\bar{f}\|_{\mathbb{L}^q_{p,\theta}(\tau)} \leqslant N\|Mv^{\ell}_{xx}\|_{\mathbb{L}^q_{p,\theta}(\tau)} \leqslant N\|M^{-1}v^{\ell}\|_{\mathbb{H}^{2,q}_{p,\theta}(\tau)} \\
&\leqslant N\|Mf^{\ell}\|_{\mathbb{L}^q_{p,\theta}(\tau)} \leqslant N\|Mf\|_{\mathbb{H}^{-1,q}_{p,\theta}(\tau)}.
\end{aligned}$$

Next, we prove the uniqueness of solutions by reducing $\bar{\beta}_0$ if necessary. Assume that $f = g^k = 0$. It is shown in [3] that there is $\beta'' = \beta''(d,p,\delta,K)$ such that if $\beta \leqslant \beta''$, then the uniqueness holds in the space $u \in \mathfrak{H}^{1,p}_{p,\theta,0}(\tau)$. Since $q \geqslant p$, $u = 0$ if we additionally assume

$$\beta \leqslant \beta' \wedge \beta''.$$

Step 2: Now we drop the additional conditions on b^i, c, v^k by showing that there exists $\bar{\beta}_0$ such that if $\beta \leqslant \bar{\beta}_0$, then the estimate (3.19) holds.

By the results of Step 1, if $\beta \leqslant \beta' \wedge \beta''$, then

$$\begin{aligned}
\|M^{-1}u\|_{\mathbb{H}^{1,q}_{p,\theta}(\tau)} &\leqslant N\|Mb^i u_{x^i} + Mcu + Mf\|_{\mathbb{H}^{-1,q}_{p,\theta}(\tau)} + \|Mvu + g\|_{\mathbb{L}^q_{p,\theta}(\tau)} \\
&\leqslant N(\|Mb^i u_{x^i} + M^2 c M^{-1}u\|_{\mathbb{L}^q_{p,\theta}(\tau)} + \|MvM^{-1}u\|_{\mathbb{L}^q_{p,\theta}(\tau)}) \\
&\quad + N\|Mf\|_{\mathbb{H}^{-1,q}_{p,\theta}(\tau)} + N\|g\|_{\mathbb{L}^q_{p,\theta}(\tau)} \\
&\leqslant N\beta(\|M^{-1}u\|_{\mathbb{L}^q_{p,\theta}(\tau)} + \|u_x\|_{\mathbb{L}^q_{p,\theta}(\tau)}) + N\|Mf\|_{\mathbb{H}^{-1,q}_{p,\theta}(\tau)} + N\|g\|_{\mathbb{L}^q_{p,\theta}(\tau)} \\
&\leqslant N_1\beta\|M^{-1}u\|_{\mathbb{H}^{1,q}_{p,\theta}(\tau)} + N\|Mf\|_{\mathbb{H}^{-1,q}_{p,\theta}(\tau)} + N\|g\|_{\mathbb{L}^q_{p,\theta}(\tau)}.
\end{aligned}$$

Thus, it suffices to additionally assume

$$\beta \leqslant 1/(2N_1).$$

The lemma is proved. □

Proof of Theorem 3.4. *Step* 1: As usual we may assume $\tau \equiv T$. By Theorem 2.12 and Step 1 in the proof of Theorem 2.10 in [5], we may assume that ∂G is infinitely differentiable. Indeed, there is $\varepsilon > 0$ and a C^{∞} diffeomorphism $\mu : G_{\varepsilon} := \{x \in G : \psi(x) > \varepsilon\} \to G$ such that

(i) the mappings μ and μ^{-1} induce one-to-one linear bounded mappings from $H^{\gamma}_{p,\theta}(G)$ onto $H^{\gamma}_{p,\theta}(G_{\varepsilon})$ and vice versa,

(ii) in the part of a neighborhood of ∂G_ε lying in G_ε we have $\psi(\mu(x)) = \psi_\varepsilon(x) := \psi(x) - \varepsilon$, and the results of (i) also hold with $\mathfrak{H}^{\gamma+2,q}_{p,\theta,0}(G,\tau)$ and $\mathfrak{H}^{\gamma+2,q}_{p,\theta,0}(G_\varepsilon,\tau)$ instead of $H^\gamma_{p,\theta}(G)$ and $H^\gamma_{p,\theta}(G_\varepsilon)$, respectively,

(iii) proving that the function $u \in \mathfrak{H}^{1,q}_{p,\theta,0}(G,\tau)$ satisfies (2.7) and admits estimate (3.14) is equivalent to proving that the the function $\tilde{u} = u(\mu) \in \mathfrak{H}^{\gamma+2,q}_{p,\theta,0}(G_\varepsilon,\tau)$ satisfies the corresponding equation in $(0,\tau] \times G_\varepsilon$, and admits the natural modification of estimate (3.14),

(iv) the mappings μ and μ^{-1} preserve the conditions of the coefficients in Assumptions 2.2 and 2.5,

(v) inequalities (3.11) and (3.12) hold if and only if the corresponding results hold for $u(v)$.

Remember that ψ is bounded and infinitely differentiable, and therefore $G_\varepsilon \in C^\infty$. Thus we conclude that we may assume that G is infinitely differentiable (see [5] for more detail).

Step 2: We establish estimates (3.14), (3.11) and (3.12) assuming that $u \in \mathfrak{H}^{1,q}_{p,\theta,0}(G,\tau)$ satisfies (2.7). Let $x_0 \in \partial G$ and Ψ be a function from Assumption 2.1. By Step 1 we may assume that Ψ is infinitely differentiable with bounded derivatives.

Define $r = r_0/K_0$ and fix a smooth function $\eta \in C_0^\infty(B_r)$ such that $0 \leqslant \eta \leqslant 1$ and $\eta = 1$ in $B_{r/2}$. Observe that $\Psi(B_{r_0}(x_0))$ contains B_r. For $m = 1,2,\dots,\ t>0,\ x \in \mathbb{R}^d_+$, introduce $\eta_m(x) = \eta(mx)$,

$$\hat{a}_m := \tilde{a}\eta_m + (1-\eta_m)I, \quad \hat{b}_m := \tilde{b}\eta_m, \quad \hat{c}_m := \tilde{c}\eta_m, \quad \hat{v}_m := \tilde{v}\eta_m,$$

where

$$\tilde{a}^{ij}(t,x) = \bar{a}^{ij}(t,\Psi^{-1}(x)), \quad \bar{a}^{ij} = a^{rs}\Psi^i_{x^r}\Psi^j_{x^s},$$

$$\tilde{b}^i(t,x) = \bar{b}^i(t,\Psi^{-1}(x)), \quad \bar{b}^i = a^{rs}\Psi^i_{x^r x^s} + b^r\Psi^i_{x^r},$$

$$\tilde{c}(t,x) = c(t,\Psi^{-1}(x)), \quad \tilde{v}(t,x) = v(t,\Psi^{-1}(x)).$$

One can easily check that there is a constant $\bar{K}'$ independent of x_0 such that

$$\sup_{m\geqslant 1}\sup_{\omega,t}(|\hat{a}_m|_1^{(0)} + |\hat{b}_m|_1^{(1)}) \leqslant \bar{K}'.$$

Take $\bar{\beta}_0$ from Lemma 3.7 corresponding to δ, p, q, K and $\bar{K}'$. By Assumption 2.5(i) and (ii) one can easily find m independent of x_0 such that

$$|\hat{a}_m(t,x) - \hat{a}_m(t,y)| + x^1|\hat{b}_m(t,x)| + (x^1)^2|\hat{c}_m(t,x)| + x^1|\hat{v}_m(t,x)|_{\ell_2} \leqslant \bar{\beta}_0,$$

whenever $\omega \in \Omega, t>0$ and $x,y \in \mathbb{R}^d_+$.

Now we fix a $\rho_0 < r_0$ such that

$$\Psi(B_{\rho_0}(x_0)) \subset B_{r/(2m)}.$$

Let ζ be a smooth function with support in $B_{\rho_0}(x_0)$ and denote $v := (u\zeta)(\Psi^{-1})$ and continue v as zero in $\mathbb{R}^d_+\backslash\Psi(B_{\rho_0}(x_0))$. Since $\eta_m = 1$ on $\Psi(B_{\rho_0}(x_0))$, the function v satisfies

$$\mathrm{d}v = (\hat{a}^{ij}_m v_{x^ix^j} + \hat{b}^i_m v_{x^i} + \hat{c}_m v + \hat{f})\,\mathrm{d}t + (\hat{v}^k_m v + \hat{g}^k)\,\mathrm{d}w^k_t,$$

where

$$\hat{f} = \tilde{f}(\Psi^{-1}), \quad \tilde{f} = -2a^{ij}u_{x^i}\zeta_{x^j} - ua^{ij}\zeta_{x^ix^j} - ub^i\zeta_{x^i} + \zeta f,$$

$$\hat{g} = \tilde{g}(\Psi^{-1}), \quad \tilde{g}^k = \zeta g^k.$$

Next we observe that by Theorem 3.2 in [12] for any $v, \alpha \in \mathbb{R}$ and $h \in \psi^{-\alpha}H^v_{p,\theta}(G)$ with support in $B_{\rho_0}(x_0)$

$$\|\psi^\alpha h\|_{H^v_{p,\theta}(G)} \sim \|M^\alpha h(\Psi^{-1})\|_{H^v_{p,\theta}}. \tag{3.24}$$

Therefore, by Lemma 3.7 we have, for any $t \leqslant T$,

$$\|M^{-1}v\|_{\mathbb{H}^{1,q}_{p,\theta}(t)} \leqslant N(\|M\hat{f}\|_{\mathbb{H}^{-1,q}_{p,\theta}(t)} + \|\hat{g}\|_{\mathbb{L}^q_{p,\theta}(t)}).$$

By using (3.24) again we obtain

$$\begin{aligned}\|\psi^{-1}u\zeta\|_{\mathbb{H}^{1,q}_{p,\theta}(G,t)} \leqslant{}& N\|a\zeta_x\psi u_x\|_{\mathbb{H}^{-1,q}_{p,\theta}(G,t)} + N\|a\zeta_{xx}\psi u\|_{\mathbb{H}^{-1,q}_{p,\theta}(G,t)}\\ &+ N\|\zeta_x\psi bu\|_{\mathbb{H}^{-1,q}_{p,\theta}(G,t)} + N\|\zeta\psi f\|_{\mathbb{H}^{-1,q}_{p,\theta}(G,t)} + N\|\zeta g\|_{\mathbb{L}^q_{p,\theta}(G,t)}.\end{aligned}$$

Also, since

$$\sup_{\omega,t}(|a\zeta_x|^{(0)}_1 + |\zeta_x\psi b|_0 + |a\zeta_{xx}\psi|_0) < \infty,$$

we conclude (see Lemma 3.15 in [5])

$$\begin{aligned}\|\psi^{-1}u\zeta\|_{\mathbb{H}^{1,q}_{p,\theta}(G,t)} \leqslant{}& N\|\psi u_x\|_{\mathbb{H}^{-1,q}_{p,\theta}(G,t)} + N\|u\|_{\mathbb{L}^q_{p,\theta}(G,t)}\\ &+ N\|\psi f\|_{\mathbb{H}^{-1,q}_{p,\theta}(G,t)} + N\|g\|_{\mathbb{L}^q_{p,\theta}(G,t)}.\end{aligned}$$

Observe that $\rho_0, m, \bar{K}', N$ are independent of x_0. Take β_0 from Lemma 3.6, which depends only on d, p, q, δ and K. To estimate the norm $\|\psi^{-1}u\|_{\mathbb{H}^{1,q}_{p,\theta}(G,t)}$, one introduces partitions of unity $\zeta_{(i)}, \xi_{(j)}, i = 1, 2, \ldots, N_1,\ j = 1, 2, \ldots, N_2$ such that $\zeta_{(i)} \in C^\infty_0(B_{\rho_0}(x_i))$, $x_i \in \partial G$, and $\xi_{(j)} \in C^\infty_0(G)$, $\operatorname{supp}\xi_{(j)} \in B_{\kappa(\beta_0)/2}(x_j)$, $x_j \in G$.

Then one estimates $\|\psi^{-1}u\xi_{(j)}\|_{\mathbb{H}^{1,q}_{p,\theta}(G,t)}$ using Lemma 3.6 and the other norms as above. By summing up those estimates one gets

$$\|\psi^{-1}u\|_{\mathbb{H}^{1,q}_{p,\theta}(G,t)} \leqslant N\|\psi u_x\|_{\mathbb{H}^{-1,q}_{p,\theta}(G,t)} + N\|u\|_{\mathbb{L}^q_{p,\theta}(G,t)} + N\|\psi f\|_{\mathbb{H}^{-1,q}_{p,\theta}(G,t)} + N\|g\|_{\mathbb{L}^q_{p,\theta}(G,t)}. \tag{3.25}$$

Furthermore, we know from Theorem 4.1 of [12] that

$$\|\psi u_x\|_{H^{-1}_{p,\theta}(G)} \leqslant N\|u\|_{L_{p,\theta}(G)}.$$

Therefore, (3.25) yields

$$\begin{aligned}\|u\|^q_{\mathfrak{H}^{1,q}_{p,\theta}(G,t)} &\leqslant N\|u\|^q_{\mathbb{L}^q_{p,\theta}(G,t)} + N\|\psi f\|^q_{\mathbb{H}^{-1,q}_{p,\theta}(G,T)} + N\|g\|^q_{\mathbb{L}^q_{p,\theta}(G,T)} \\ &\leqslant N\int_0^t \|u\|^q_{\mathfrak{H}^{1,q}_{p,\theta}(G,s)}\,\mathrm{d}s + N\|\psi f\|^q_{\mathbb{H}^{-1,q}_{p,\theta}(G,t)} + N\|g\|^q_{\mathbb{L}^q_{p,\theta}(G,t)},\end{aligned} \tag{3.26}$$

where the second inequality can be obtained from Theorem 4.11 and Corollary 4.12 in [9]. Indeed, denote $\zeta_{(0)} = \sum \xi_{(j)}$, then

$$E\sup_{t\leqslant T}\|u\|^q_{L_{p,\theta}(G)} \leqslant NE\sup_{t\leqslant T}\|u\zeta_{(0)}\|^q_{L_{p,\theta}(G)} + N\sum_{i\geqslant 1} E\sup_{t\leqslant T}\|u\zeta_{(i)}\|^q_{L_{p,\theta}(G)}.$$

Let Ψ be a function from Assumption 2.1 corresponding to x_i and denote $v_i = \zeta_{(i)}u$ and $\bar{v}_i := v_i(\Psi^{-1})$, then by (3.24) and Theorem 4.1 in [9] (see inequality (4.5) there), for $i\geqslant 1$,

$$E\sup_{t\leqslant T}\|v_i\|^q_{L_{p,\theta}(G)} \leqslant NE\sup_{t\leqslant T}\|\bar{v}_i\|^q_{L_{p,\theta}} \leqslant N\|\bar{v}_i\|^q_{\mathfrak{H}^{1,q}_{p,\theta}(T)} \leqslant N\|v_i\|^q_{\mathfrak{H}^{1,q}_{p,\theta}(G,T)}.$$

It is easy to check (see, for instance, Proposition 2.1 in [12]) that if $h \in H^\gamma_{p,\theta}(G)$ has compact support in G, then $h \in H^\gamma_p$ and

$$\|h\|_{H^\gamma_{p,\theta}(G)} \sim \|h\|_{H^\gamma_p}. \tag{3.27}$$

Thus, by Corollary 4.12 in [9] (see inequality (4.18) there),

$$\begin{aligned}E\sup_{t\leqslant T}\|u\zeta_{(0)}\|^q_{L_{p,\theta}(G)} &\leqslant NE\sup_{t\leqslant T}\|v_0\|^q_{L_p} \\ &\leqslant N\|v_0\|^q_{\mathscr{H}^{1,q}_p(T)} \leqslant N\|v_0\|^q_{\mathfrak{H}^{1,q}_{p,\theta}(G,T)}.\end{aligned}$$

Consequently,

$$E\sup_{t\leqslant T}\|u\|^q_{L_{p,\theta}(G)} \leqslant N\|u\|^q_{\mathfrak{H}^{1,q}_{p,\theta}(G,T)}$$

and in particular,

$$\|u\|^q_{\mathbb{L}^q_{p,\theta}(G,t)} \leqslant N\int_0^t \|u\|^q_{\mathfrak{H}^{1,q}_{p,\theta}(G,s)}\,\mathrm{d}s. \tag{3.28}$$

Now (3.14) follows from inequality (3.26) and Gronwall's inequality.

Step 3: Finally, considering the method of continuity, we finish the proof by showing that for any $(f,g) \in \psi^{-1}\mathbb{H}^{-1,q}_{p,\theta}(G,T)\times\mathbb{L}^q_{p,\theta}(G,T)$, there exists $u \in$

$\mathfrak{H}^{1,q}_{p,\theta,0}(G,T)$ such that

$$\mathrm{d}u = (\Delta u + f)\,\mathrm{d}t + g^k\,\mathrm{d}w^k_t, \quad u(0,\cdot) = 0. \tag{3.29}$$

We can approximate $g = (g^1, g^2, \ldots)$ with functions having only finite nonzero entries and it is known (see [12]) that smooth functions with compact support are dense in $H^{\nu}_{p,\theta}(G)$. Therefore, it follows from a priori estimate (3.14) that we may assume that g has only finite nonzero entries and is bounded on $\Omega \times [0,T] \times G$ along with each derivative in x and vanishes if x is near ∂G. In that case it is well known that

$$v(t,x) := \int_0^t g^k(t,x)\,\mathrm{d}w^k_s$$

is infinitely differentiable in x and vanishes near ∂G. Therefore, we conclude $v \in \mathfrak{H}^{\nu,q}_{p,\theta}(G,T)$ for any $\nu \in \mathbb{R}$. Observe that Eq. (3.29) can be written as

$$\mathrm{d}\bar{u} = (\Delta \bar{u} + f + \Delta v)\,\mathrm{d}t,$$

where $\bar{u} := u - v$. Thus, we reduced the case to the case in which $g \equiv 0$. So we may assume $g = 0$ in (3.29). The same arguments show that we may assume that f is bounded on $\Omega \times [0,T] \times G$ along with each derivative in (t,x) and vanish if x is in a neighborhood of ∂G.

Now let T_t be the heat semigroup corresponding to the heat equation on G with zero boundary condition. Define

$$u(t,x) = \int_0^t T_{t-s} f(s,x)\,\mathrm{d}s.$$

It is well known (see [11]) that u is the classical solution of the heat equation with zero boundary and initial data, and $u/\psi, u_x$ are bounded in $\Omega \times [0,T] \times G$. Thus, we conclude $u \in \mathfrak{H}^{1,q}_{p,\theta,0}(G,T)$.

The theorem is proved. $\square$

Proof of Lemma 3.1. We will use the notations in the above proof. To prove this lemma it is enough to combine Theorems 4.1 and 4.11 in [9]. Indeed,

$$E|\psi^{\beta-1}u|^q_{C^{\alpha/2-1/q}([0,\tau],H^{\gamma-\beta}_{p,\theta}(G))} \leqslant N \sum_{i \geqslant 0} E|\psi^{\beta-1}u\zeta_{(i)}|^q_{C^{\alpha/2-1/q}([0,\tau],H^{\gamma-\beta}_{p,\theta}(G))}.$$

We can replace the function $\psi^{\beta-1}$ with a $C^{\infty}_0(\mathbb{R}^d)$-function equal to $\psi^{\beta-1}$ on the support of $u\zeta_{(0)}$, and therefore we conclude

$$\|\psi^{\beta-1}u\zeta_{(0)}\|_{H^{\gamma-\beta}_p} \leqslant N\|u\zeta_{(0)}\|_{H^{\gamma-\beta}_p}.$$

Thus, by Theorem 4.11 in [9] and (3.27),

$$\begin{aligned} E|\psi^{\beta-1}u\zeta_{(0)}|^q_{C^{\alpha/2-1/q}([0,\tau],H^{\gamma-\beta}_{p,\theta}(G))} &\leqslant NE|\psi^{\beta-1}u\zeta_{(0)}|^q_{C^{\alpha/2-1/q}([0,\tau],H^{\gamma-\beta}_p)} \\ &\leqslant NE|u\zeta_{(0)}|^q_{C^{\alpha/2-1/q}([0,\tau],H^{\gamma-\beta}_p)} \leqslant N\|u\zeta_{(0)}\|^q_{\mathscr{H}^{1,q}_p(\tau)} \\ &\leqslant N\|u\zeta_{(0)}\|^q_{\mathfrak{H}^{1,q}_{p,\theta}(G,\tau)}. \end{aligned}$$

Similarly for $i \geqslant 1$, by (3.24) and Theorem 4.1 in [9],

$$\begin{aligned} E|\psi^{\beta-1} u\zeta_{(i)}|^q_{C^{\alpha/2-1/q}([0,\tau],H^{\gamma-\beta}_{p,\theta}(G))} &\leqslant E|M^{\beta-1}\bar{v}_i|^q_{C^{\alpha/2-1/q}([0,\tau],H^{\gamma-\beta}_{p,\theta})} \\ &\leqslant N\|\bar{v}_i\|^q_{\mathfrak{H}^{\gamma,q}_{p,\theta}(\tau)} \\ &\leqslant N\|v_i\|^q_{\mathfrak{H}^{\gamma,q}_{p,\theta}(G,\tau)} = \|u\zeta_{(i)}\|^q_{\mathfrak{H}^{\gamma,q}_{p,\theta}(G,\tau)}. \end{aligned}$$

Thus, (3.8) is proved. One can prove (3.7) similarly. The lemma is proved. □

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